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Hyperelliptic Curves Describing Coulomb Phase of $N = 2$ Supersymmetric Theories with Classical Gauge Groups $SU(3)$ and $SO(6)$

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Breaking $N = 2$ $SU(3)$ and $N = 2$ $SO(6)$ supersymmetric Yang–Mills theories to corresponding $N = 1$ theories by suitable tree-level superpotentials, the hyperelliptic curves describing the Coulomb phase of these theories have been obtained and it has been shown that the mass gap in the $N = 1$ confining phase of these theories vanishes when $N = 1$ parameters are properly tuned to approach the highest critical points.

1. INTRODUCTION

In two remarkable papers Seiberg and Witten [1, 2] obtained exact information on the dynamics of $N = 2$ supersymmetric gauge theories in four dimensions with the gauge group $SU(2)$ and demonstrated that the strongly coupled vacuum turns out to be the weakly coupled theory of monopoles. Following this work, much progress has been made in understanding the four-dimensional $N = 2$ supersymmetric gauge theories. Recently, we have undertaken [3] the study of monopoles and dyons in four-dimensional $N = 2$ supersymmetric theory with gauge group $SU(2)$, carried out [4] the analysis of kinematics of moduli space vacua, and obtained [5] the spectrum of BPS states of dyons in weak- and strong-coupling regions. A crucial advantage of using $N = 2$ supersymmetry is that the low-energy effective action in the Coulomb phase up to two derivatives is determined in terms of a single function (i.e., prepotential) [6]. The dynamics of $N = 1$ supersymmetric

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gauge theories of monopoles and dyons in four dimensions has been thoroughly explored [7, 8] and exact results have been obtained [9–11] about their coupling behavior by using holomorphic properties of the superpotential and gauge kinetic function culminating in Seiberg's non-Abelian duality conjecture [10, 12]. In all these exact solutions, the singularities of quantum moduli space of the theory correspond to the appearance of massless dyons. Consequently, a microscopic superpotential explicitly breaking $N = 2$ to $N = 1$ supersymmetry has been introduced [13, 14] to explore physics near $N = 2$ singularities, and it has been found that the generic $N = 2$ vacuum is lifted, leaving only a singular locus of moduli space as the $N = 1$ vacua where monopoles and dyons can condense. By soft breaking of $N = 2$ down to $N = 1$, confinement follows due to monopole condensation.

Perturbing an $N = 2$ theory by adding a tree-level superpotential, one can get [15] a microscopic $N = 1$ theory where it is convenient to concentrate on a phase with a single confined photon. Then the low-energy effective theory, containing nonperturbative effects, provides us with the data of the vacua with massless dyons [16]. For the $N = 2$ theory broken to $N = 1$, two different Lagrangians have been constructed [2]. Both these lead to the same physics for the massless modes and differ only in the way they describe the massive fields. In the present paper, we break $N = 2$ $SU(3)$ and $N = 2$ $SO(6)$ supersymmetric Yang–Mills theories to corresponding $N = 1$ theories on perturbing these theories by suitable tree-level superpotentials and obtain the effective superpotentials for the phase with a confined photon in $N = 1$ supersymmetric gauge theories. We also derive the hyperelliptic curves which describe the Coulomb phase of $N = 2$ theories with classical gauge groups $SU(3)$ and $SO(6)$ and demonstrate how the microscopic parameters in $N = 1$ theory are related to the $N = 2$ moduli coordinates. It is shown that the mass gap of $N = 1$ theory due to dyon condensation vanishes as we approach the Z_3 critical point in $N = 2$ $SU(3)$ theory. It is also shown that the $N = 1$ mass gap vanishes at a singular point of $N = 2$ $SO(6)$ theory where the single massless dyon exists. It is demonstrated how to derive the curves for the Coulomb phase of these $N = 2$ Yang–Mills theories with classical gauge groups $SU(3)$ and $SO(5)$ by means of $N = 1$ confining phase superpotential.

2. BREAKING OF $N = 2$ $SU(3)$ SUPERSYMMETRY

Let us start with $N = 2$ $SU(3)$ Yang–Mills theory and perturb it [15, 17] by a tree-level superpotential,

$$W = g_1 u_1 + g_2 u_2 + g_3 u_3 \tag{2.1}$$

with

$$u_1 = \text{tr}\ \phi; \qquad u_2 = \frac{1}{2}\text{tr}\ \phi^2 \quad ; \qquad u_3 = \frac{1}{3}\text{tr}\ \phi^3 \tag{2.1a}$$

where ϕ is an adjoint $N = 1$ superfield in the $N = 2$ vector multiplet and g_1 is an auxiliary field implementing $\text{tr}\phi = 0$. Thus we have $u_1 = 0$, and u_2 parametrises the classical moduli space [3, 5] of dyons when ϕ is the Higgs field in the supersymmetric theory of dyons. The classical vacuum of the theory is determined by the equation of motion

$$W'(\phi) = 0$$

which leads to

$$g_1 + g_2\phi + g_3\phi^2 = 0 \tag{2.2}$$

or

$$\phi = \frac{-g_2 \pm \sqrt{g_2^2 - 4g_1g_3}}{2g_3} \tag{2.2a}$$

These eigenvalues of ϕ are the roots of the equation

$$W'(x) = g_3(x - a_1)(x - a_2) \tag{2.3}$$

where we have set

$$g_1 = a_1a_2g_3 \tag{2.4}$$

$$g_2 = (a_1 + a_2)g_3$$

Substituting these relations in (2.2a), we get the following eigenvalues of ϕ:

$$\phi = a_1, a_1, a_2$$

or

$$\phi = \text{diag}(a_1, a_1, a_2) \tag{2.5}$$

with

$$a_2 = -2a_1 \qquad \text{for} \quad \text{tr}\ \phi = 0$$

This ϕ describes the unbroken $SU(2) \times U(1)$ vacuum. In the low-energy limit the adjoint superfield for $SU(2)$ becomes massive and it will be decoupled. We are then left with an $N = 1$ $SU(2)$ Yang–Mills theory which is in the confining phase and the photon multiplets for $U(1)$ are decoupled.

In order to obtain the relation between the $SU(3)$ scale Λ and the low-energy $SU(2)$ scale Λ_L, we first match at the scale $SU(3)/SU(2)$ the W-boson and then match at $SU(2)$ the adjoint mass M_d [18]. We get

$$\Lambda^6 = \Lambda_L^6(a_1 - a_2)^2 M_{ad}^{-2} \tag{2.6}$$

Let us decompose ϕ in the following manner:

$$\phi = \phi_{cl} + \delta\phi + \delta\overline{\phi} \tag{2.7}$$

where ϕ_{cl} is given by (2.5); $\delta\phi$ denotes the fluctuation along the unbroken $SU(2)$ direction and $\delta\overline{\phi}$ is the fluctuation along other directions. Substituting this result into Eq. (2.1), we get

$$W = W_{cl} + \frac{1}{2} g_3(a_1 - a_2) \operatorname{tr} \delta\phi^2 \tag{2.8}$$

where

$$(\delta\phi, \phi_{cl}) = 0$$

and W_{cl} is the tree-level superpotential evaluated in the classical vacuum.

Thus we get

$$M_{ad} = g_3(a_1 - a_2) = W'(a_1) \tag{2.9}$$

Substituting it into (2.6), we get

$$\Lambda_L^6 = g_3^2\Lambda^6 \qquad \text{or} \qquad \frac{\Lambda_L}{\Lambda} = (g_3)^{1/3} \tag{2.10}$$

Starting with $N = 2$ $SU(N_c)$ Yang–Mills theory and perturbing it by a suitable tree-level superpotential, we got the following generalizations of (2.10):

$$\begin{aligned} \Lambda_L/\Lambda^2 &= (g_6)^{1/3} && \text{for} \quad N_c = 6 \\ \Lambda_L/\Lambda^3 &= (g_9)^{1/3} && \text{for} \quad N_c = 9 \\ \Lambda_L/\Lambda^4 &= (g_{12})^{1/3} && \text{for} \quad N_c = 12 \\ \Lambda_L/\Lambda^5 &= (g_{15})^{1/3} && \text{for} \quad N_c = 15 \end{aligned} \tag{2.10a}$$

or in general for $N_c = 3n$.

$$\Lambda_L/\Lambda^n = (g_{N_c})^{1/3} \tag{2.10b}$$

But the gaugino condensation dynamically generates the superpotential in the $N = 1$ $SU(2)$ theory, and hence the low-energy effective superpotential takes the form [18]

$$W_L = W_{cl} \pm 2\,\Lambda_L^3 \tag{2.11}$$

which reduces to the following form for equation (2.10);

$$W_L = W_{\rm cl} \pm 2g_3\,\Lambda^3 \tag{2.12}$$

It is exact for any values of the parameters. From this equation we get

$$\begin{aligned}
\langle u_1 \rangle &= \frac{\partial W_L}{\partial g_1} = \frac{\partial W_{\rm cl}}{\partial g_1} = u_{1\rm cl} \\
\langle u_2 \rangle &= \frac{\partial W_L}{\partial g_2} = \frac{\partial W_{\rm cl}}{\partial g_2} = u_{2\rm cl} \\
\langle u_3 \rangle &= \frac{\partial W_L}{\partial g_3} = \frac{\partial W_{\rm cl}}{\partial g_3} \pm 2\Lambda^3 = u_{3\rm cl} \pm 2\Lambda^3
\end{aligned} \tag{2.13}$$

where $u_{n\rm cl} = \partial W_{\rm cl}/\partial g_n$ (for $n = 1, 2, 3$) are the classical values of u_n as given by Eqs. (2.1a). These vacua correspond to the singular loci of $N = 2$ massless dyons. To check this, we plug these results into the $N = 2$ $SU(3)$ curve [19, 20],

$$y^2 = \langle \det(x - \phi) \rangle^2 - 4\Lambda^6$$

or

$$y^2 = [x^3 - \langle s_2 \rangle x + \langle s_3 \rangle]^2 - 4\Lambda^6 \tag{2.14}$$

where $s_2 = u_2$ and $s_3 = u_3$

Substituting relations (2.13) into Eq. (2.14), we get

$$\begin{aligned}
y^2 &= [x^3 - \langle u_2 \rangle x + \langle u_3 \rangle]^2 - 4\Lambda^6 \\
&= (x^3 - u_{2\rm cl}x - u_{3\rm cl})(x^3 - u_{2\rm cl}x - u_{3\rm cl} \pm 4\Lambda^3)
\end{aligned} \tag{2.15}$$

Using Eqs. (2.1a) and (2.5), we have

$$u_{2\rm cl} = \frac{1}{2}\,\mathrm{tr}\,\phi^2 = 3a_1^2$$

and

$$u_{3\rm cl} = \frac{1}{3}\,\mathrm{tr}\,\phi^3 = -2a_1^3$$

Substituting these relations into Eq. (2.15), we get

$$y^2 = (x - a_1)^2(x - a_2)[(x - a_1)^2(x - a_2) \pm 4\Lambda^3] \tag{2.16}$$

This curve exhibits the quadratic degeneracy and hence we are exactly at the singular point of a massless dyon in the $N = 2$ $SU(3)$ Yang–Mills vacuum.

In $N = 2$ $SU(3)$ theory the $N = 2$ highest critical points [21] exist at $\langle u_2 \rangle = 0$ and $\langle u_3 \rangle = \pm 2\Lambda^3$. These critical points feature by Z_3 symmetry.

When we approach these points under $N = 1$ perturbation, the coupling constants of Eq. (2.1) become

$$g_2 \to 0 \tag{2.17}$$

and then Eq. (2.1) reduces to

$$W_{\text{crit}} = g_3 u_3 = -\frac{2}{3} g_3 a_1^3 \tag{2.18}$$

In $N = 1$ theory there exists a mass gap due to dyon condensation and the gauge fields get a mass by the magnetic Higgs mechanism. In order to check the behavior of this gap in the limit (2.17), let us consider a macroscopic $N = 1$ superpotential W_m obtained from the effective low-energy Abelian theory. Let us denote the $N= 1$ chiral superfield of $N = 2U(1)$ multiplets by A_1, and $N = 1$ chiral superfields of $N = 2$ dyon hypermultiplets by M_1 and M_1'. Then we have[14]

$$W_m = \sqrt{2}[A_1 M_1 M_1' + A_2 M_2 M_2'] + g_2 U_2 + g_3 U_3 \tag{2.19}$$

where U_2 and U_3 represent the superfields corresponding to tr ϕ^2 and tr ϕ^3, respectively, with their lowest components having expectation values $\langle u_2 \rangle$ and $\langle u_3 \rangle$. Then the equations of motion are given by

$$\begin{aligned}
\frac{-g_2}{\sqrt{2}} &= \frac{\partial \alpha_1}{\partial u_2} m_1 m_1' + \frac{\partial \alpha_2}{\partial u_2} m_2 m_2' \\
\frac{-g_3}{\sqrt{2}} &= \frac{\partial \alpha_1}{\partial u_3} m_1 m_1' + \frac{\partial \alpha_2}{\partial u_3} m_2 m_2' \\
\alpha_1 m_1 &= \alpha_1 m_1' = 0 \\
\alpha_2 m_2 &= \alpha_2 m_2' = 0
\end{aligned} \tag{2.20}$$

where α_1 and α_2 are expectation values of the lowest components of A_1 and A_2; m_1 and m_2 are expectation values of the lowest components of M_1 and M_2; and m_1' and m_2' are the expectation values of the lowest components of M_1' and M_2'.

The D-flatness condition [22] implies that

$$|m_1| = |m_1'|, \qquad |m_2| = |m_2'| \tag{2.21}$$

Let us consider a singular point where we have only one massless dyon M_1, M_1'. Then $\alpha_1 = 0$ and $\alpha_2 \neq 0$, and Eqs. (2.20) lead to

$$\begin{aligned}
m_2 &= 0 \\
-\frac{g_2}{\sqrt{2}} &= \frac{\partial \alpha_1}{\partial u_2} m_1 m_1'
\end{aligned} \tag{2.22}$$

$$-\frac{g_3}{\sqrt{2}} = \frac{\partial \alpha_1}{\partial u_3} m_1 m_1'$$

which give

$$\frac{g_2}{g_3} = \frac{\partial \alpha_1/\partial \langle u_2 \rangle}{\partial \alpha_1/\partial \langle u_3 \rangle} = \frac{\partial \langle u_3 \rangle}{\partial \langle u_2 \rangle} \tag{2.23}$$

Let us bring the system to Z_3-critical points by tuning a parameter ϵ such that

$$\langle u_2 \rangle = c_2 \epsilon^2, \qquad \langle u_3 \rangle = c_3 \epsilon^3 \pm 2\Lambda^3 \tag{2.24}$$

where ϵ is an overall mass scale and c_2 and c_3 are constants. These relations show that as $\epsilon \to 0$ we have

$$\langle u_2 \rangle = 0, \qquad \langle u_3 \rangle = \pm 2\Lambda^3$$

i.e., we are at Z_3-critical points.

From Eqs. (2.24), we have

$$\frac{\partial \langle u_3 \rangle}{\partial \langle u_2 \rangle} \rightsquigarrow \epsilon \tag{2.25}$$

Substituting this result into Eq. (2.25), we get

$$\frac{g_2}{g_3} \rightsquigarrow \epsilon$$

showing that $g_2 \to 0$ as $\epsilon \to 0$. This agrees with relation (2.17). From Eqs. (2.22), we have the scaling behavior

$$m_1 = \left(\frac{-g_3}{\sqrt{2}\ \partial \alpha_1/\partial \langle u_3 \rangle} \right)^{1/2} \tag{2.26}$$

Following Argyres and Douglas [14] and Eguchi *et al.* [21], we have

$$\frac{\partial \alpha_1}{\partial \langle u_2 \rangle} = \epsilon^{3/2}, \qquad \frac{\partial \alpha_1}{\partial \langle u_3 \rangle} = \epsilon^{-1/2} \tag{2.27}$$

Substituting Eq. (2.27) into Eq. (2.26), we get

$$m_1 = \left(-\frac{g_3}{\sqrt{2}} \right)^{1/2} \epsilon^{1/4} \tag{2.28}$$

showing that

$$m_1 \to 0 \qquad \text{as} \quad \epsilon \to 0$$

Thus the mass gap due to dyon condensation vanishes as we approach the

Z_3-critical point in our theory. This shows that the Z_3 vacuum of $N = 1$ theory, characterized by the superpotential given by Eq. (2.18), is a nontrivial fixed point.

3. BREAKING OF $N = 2$ $SO(6)$ SUPERSYMMETRY

In this section we start with $N = 2$ $SO(6)$ Yang–Mills theory and perturb it by the following tree-level superpotential, which breaks $N = 2$ to $N = 1$:

$$W = g_2 u_2 + g_4 u_4 + \lambda v \tag{3.1}$$

where

$$\left.\begin{aligned} u_2 &= \frac{1}{2}\,\mathrm{tr}\,\phi^2; \qquad u_4 = \frac{1}{4}\,\mathrm{tr}\,\phi^4 \\ v &= \frac{1}{48}\,\epsilon_{i_1 i_2 j_1 j_2 k_1 k_2}\phi^{i_1 i_2}\phi^{j_1 j_2}\phi^{k_1 k_2} \\ &= \text{Pfaffian}\ \phi = P_f\phi \end{aligned}\right\} \tag{3.2}$$

with the adjoint superfield ϕ as an antisymmetric 6×6 matrix. The theory has classical vacua (i.e., moduli space) which satisfy the condition

$$W'(\phi) = 0$$

or

$$[W'(\phi)]_{ij} = g_2\phi_{ij} + g_4\phi^3_{ij} - \frac{\lambda}{16}\,\epsilon_{iji_1 i_2 j_1 j_2}\phi^{i_1 i_2}\phi^{j_1 j_2} = 0 \tag{3.3}$$

We choose the following skew-diagonal form of ϕ:

$$\phi = \mathrm{diag}(\sigma_2 e_0, \sigma_2 e_1, \sigma_2 e_2) \tag{3.4}$$

where

$$\sigma_2 = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}$$

Then the vacuum condition (3.3) leads to

$$\left.\begin{aligned} g_2 e_0^2 + g_4 e_0^4 + \frac{i\lambda}{2}\,e_0 e_1 e_2 &= 0 \\ g_2 e_1^2 + g_4 e_1^4 + \frac{i\lambda}{2}\,e_0 e_1 e_2 &= 0 \\ g_2 e_2^2 + g_4 e_2^4 + \frac{i\lambda}{2}\,e_0 e_1 e_2 &= 0 \end{aligned}\right\} \tag{3.5}$$

showing that nonvanishing e_0, e_1, and e_2 are the roots of

$$f(x) = g_2 x^2 + g_4 x^4 + \frac{i\lambda}{2} e_0 e_1 e_2 = 0 \tag{3.6}$$

Concentrating on the unbroken $SU(2) \times U(1) \times U(1)$ vacuum with a single confined photon, we may write Eq. (3.6) in the form

$$f(x) = g_4(x^2 - a_1^2)(x^2 - a_2^2) = 0 \tag{3.7}$$

where

$$g_4 a_1^2 a_2^2 = \frac{i\lambda}{2} e_0 e_1 e_2 \qquad \text{and} \qquad g_2 = -g_4(a_1^2 + a_2^2) \tag{3.8}$$

Equations (3.5) and (3.8) lead to

$$a_2 = \frac{i\lambda}{2g_4}; \qquad a_1 = \frac{\sqrt{\lambda^2 - 4g_2 g_4}}{2g_4} \tag{3.9}$$

$$e_0 = e_1 = a_1; \qquad e_2 = a_2$$

Substituting these values into Eq. (3.4), we get

$$\phi = \text{diag}(\sigma_2 a_1, \sigma_2 a_1, \sigma_2 a_2) \tag{3.10}$$

which is obviously a traceless 6×6 matrix.

Making the scale matching between the $SO(6)$ scale Λ and the $SU(2)$ scale Λ_L by following similar steps as taken in the $SU(3)$ case in the last section, we get

$$\Lambda^8 = \Lambda_L^6 (a_1^2 - a_2^2)^2 (M_{ad})^{-2} \tag{3.11}$$

where the factor arising through the Higgs mechanism is calculated in an explicit basis of $SO(6)$.

For evaluating the $SU(2)$ adjoint mass M_{ad}, let us substitute the decomposition given by Eq. (2.7) into Eq. (3.1). Then we have

$$W = W_{\text{cl}} + \frac{g_1}{2} \text{tr}(\delta\phi^2) + \frac{3g_2}{2} \text{tr}(\delta\phi^2 \phi_{\text{cl}}^2) + \frac{\lambda}{4} (\text{tr } \delta\phi^2)(-ia_2)$$

or

$$W = W_{\text{cl}} + \frac{1}{2} \frac{d}{dx} \left[\frac{f(x)}{x} \right] \text{tr } \delta\phi^2$$
$$= W_{\text{cl}} + g_4(a_1^2 - a_2^2) \text{ tr } \delta\phi^2 \tag{3.12}$$

which leads to the result

$$M_{ad} = g_4(a_1^2 - a_2^2) \tag{3.13}$$

Substituting this relation into Eq. (3.11), we get

$$\Lambda^8 = \Lambda_L^6/g_4^2 \qquad \text{or} \qquad \Lambda_L^3 = g_4\Lambda^4 \tag{3.14}$$

The low-energy superpotential thus becomes

$$W_L = W_{\rm cl} \pm 2\,\Lambda_L^3 = W_{\rm cl} \pm 2g_4\Lambda^4 \tag{3.15}$$

where the second term is due to gaugino condensation in the low-energy $SU(2)$ theory. From Eq. (3.15) we get the following vacuum expectation values of gauge invariants:

$$\begin{aligned} \langle u_2\rangle &= \frac{\partial W_L}{\partial g_2} = u_{2\rm cl} \\ \langle u_4\rangle &= \frac{\partial W_L}{\partial g_4} = u_{4\rm cl} \pm 2\Lambda^4 \\ \langle v\rangle &= \frac{\partial W_L}{\partial_\lambda} = v_{\rm cl} \end{aligned} \tag{3.16}$$

Following the approach of Brandhuber and Landsteiner [23] and also that of Terashima and Yang [15], we get the following curve for our $N = 2$ $SO(6)$ theory:

$$y^2 = \langle \det(x - \phi)\rangle^2 - 4\Lambda^8 x^4 \tag{3.17}$$

This equation may also be written as

$$y^2 = [x^6 - \langle s_2\rangle x^4 - \langle s_4\rangle x^2 - \langle v\rangle^2]^2 - 4\Lambda^8 x^4 \tag{3.18}$$

where

$$\left.\begin{aligned} s_2 &= -\frac{u_2^2}{2} + u_4 \\ s_4 &= -\frac{u_2^4}{24} - u_2u_6 + \frac{1}{2}\,u_2^2u_4 - \frac{u_4^2}{2} + u_8 \end{aligned}\right\} \tag{3.19}$$

Using relations (3.16), (3.2), and (3.4) in these equations, we get

$$\begin{aligned} \langle s_2\rangle &= -\frac{1}{2}\,\langle u_2{\rm cl}\rangle^2 + \langle u_4{\rm cl}\rangle \pm 2\Lambda^4 \\ &= -a_1^4 - 2a_1^2a_2^2 \pm 2\Lambda^4 \end{aligned} \tag{3.20}$$

$$\begin{aligned} \langle s_4\rangle &= \pm 2\Lambda^4\{a_1^4 + 2a_1^2a_2^2 \mp 2\Lambda^4 \pm \Lambda^4\} \\ &= \pm 2\Lambda^4\{-\langle s_2\rangle \pm \Lambda^4\} \end{aligned} \tag{3.21}$$

Using Eqs. (3.2) and (3.4), we also get

$$\langle v \rangle = a_1^2 a_2 \tag{3.22}$$

Substituting relations (3.20)–(3.22) into Eq. (3.18), we immediately observe the quadratic degeneracy

$$y^2 \approx (x^2 - a_1^2)^2 (x^2 - a_2^2) \tag{3.22a}$$

in the curve for $N = 2$ $SO(6)$. It is also obvious that the apparent singularity at $\langle v \rangle = 0$ is not realized in the resulting $N = 1$ theory. In this case the curve (3.18) reduces to

$$\begin{aligned} y^2 = x^4[&x^4 - \langle s_2 \rangle x^2 + 2\Lambda^4\{\pm\langle s_2 \rangle + 1 - \Lambda^4\}] \\ &\times [x^4 - \langle s_2 \rangle x^2 + 2\Lambda^4\{\pm\langle s_2 \rangle - 1 - \Lambda^4\}] \end{aligned} \tag{3.23}$$

Thus the point $\langle v \rangle = 0$ does not correspond to massless solutions

The $N = 2$ $SO(6)$ theory possesses the highest critical points

$$\langle u_2 \rangle = 0, \qquad \langle v \rangle = 0, \qquad \langle u_4 \rangle = \pm 2\Lambda^4 \tag{3.24}$$

Then

$$\langle s_2 \rangle = \pm 2\Lambda^4 \qquad \text{and} \qquad \langle s_4 \rangle = \mp 2\Lambda^8$$

and hence the equation of curve (3.23) reduces to

$$y^2 = x^4[x^4 - 2\Lambda^4(x^2 + 1) + 2\Lambda^8][x^4 - 2\Lambda^4(x^2 - 1) + 2\Lambda^8] \tag{3.25}$$

In the $N = 1$ superpotential (3.1) this critical condition corresponds to

$$g_2 \to 0, \qquad \lambda \to 0 \tag{3.26}$$

and

$$\begin{aligned} W_{\text{crit}} - g_4 u_4 &= \frac{g_4}{4} \operatorname{tr} \phi^4 \\ &= g_4\left(a_1^4 + \frac{a_2^4}{2}\right) \end{aligned} \tag{3.27}$$

Let us now look at the singular point where a single massless dyon exists. The vacuum condition in this case may be written as

$$\frac{g_2}{g_4} = \frac{\partial\alpha_1/\partial\langle u_2 \rangle}{\partial\alpha_1/\partial\langle u_4 \rangle} = \frac{\partial\langle u_4 \rangle}{\partial\langle u_2 \rangle} \tag{3.28}$$

and

$$\frac{\lambda}{g_4} = \frac{\partial\alpha_1/\partial\langle v \rangle}{\partial\alpha_1/\partial\langle u_4 \rangle} = \frac{\partial\langle u_4 \rangle}{\partial\langle v \rangle}$$

With the parametrization

$$\begin{aligned} \langle u_2 \rangle &= c_1 \epsilon^2 \\ \langle u_4 \rangle &= c_2 \epsilon^4 \mp 2\Lambda^4 \\ \langle v \rangle &= c\epsilon^3 \end{aligned} \tag{3.29}$$

where ϵ is an overall mass scale, and c_1, c_2, c_3 are constants, relations (3.28) yield

$$\frac{g_2}{g_4} \approx \epsilon^2 \to 0 \qquad \text{and} \qquad \frac{\lambda}{g_4} \approx \epsilon \to 0 \tag{3.29a}$$

which are in agreement with Eqs. (3.26). In this limit the gap in the $U(1)$ factor scale is

$$\begin{aligned} m_1 &= \left(\frac{-g_4}{\sqrt{2}\ \partial\alpha_1/\partial\langle u_4 \rangle} \right)^{1/2} \\ &\approx \sqrt{g_4}\, \epsilon^{1/2} \to 0 \end{aligned} \tag{3.30}$$

Thus the $N = 1$ gap vanishes at the singular point where a single massless dyon exists. In other words, the $N = 1$ $SO(6)$ theory with the superpotential (3.27) has a nontrivial fixed point.

4. DISCUSSION

In the Coulomb phase of $N = 2$ $SU(3)$ Yang–Mills theory the gauge symmetry breaks down to $U(1) \times U(1)$. Near the singularity of a massless dyon we have a photon coupled to the light dyon hypermultiplets, while the photon for the $U(1)$ factor remains free. The tree-level superpotential (2.1) perturbs this theory and we are left with an $N = 1$ $SU(2)$ Yang–Mills theory described by a Higgs field given by (2.5), which is in the confining phase, and the photon multiplets for the $U(1)$ factor are decoupled. Equation (2.10) gives the relationship between the $SU(3)$ scale Λ and the low-energy $SU(2)$ scale Λ_L. Equations (2.10a) and (2.10b) are the generalizations of this relation for the cases of $SU(6)$, $SU(9)$, $SU(12)$, $SU(15)$, and the most general case of $SU(3n)$. Equations (2.13) describe the vacua corresponding to the singular loci of $N = 2$ massless dyons, and the quadratic degeneracy in the curve (2.16) shows that we are exactly at the singular point of a massless dyon in the $N = 2$ $SU(3)$ Yang–Mills vacuum. In this approach we can explicitly read off how the microscopic parameters in $N = 1$ theory are related to the $N = 2$ moduli coordinates. Equation (2.28) shows that the mass gap of $N = 1$ theory due to dyon condensation vanishes as we approach the Z_3 critical point in $N = 2$ $SU(3)$ theory. Thus the Z_3 vacuum of $N = 1$ theory characterized by the superpotential given by Eq. (2.18) is a nontrivial fixed point.

The tree-level potential (3.1) breaks the $N = 2$ $SO(6)$ Yang–Mills theory to $N = 1$ theory, leaving the unbroken $SU(2) \times U(1) \times U(1)$ vacuum with a single confined photon. The scale matching between the $SO(6)$ scale Λ and the $SU(2)$ scale Λ_L is given by Eq. (3.14) with the low-energy superpotential given by Eq. (3.15), which leads to the vacuum expectation values of gauge invariants as given by Eqs. (3.16). The hyperelliptic curve for $N = 2$ $SO(6)$ theory is given by Eq. (3.23), showing the quadratic degeneracy (3.22a). At the highest critical point, given by Eqs. (3.24) for $N = 2$ $SO(6)$ theory, the equation of curve reduces to the form of Eq. (3.25). This criticality corresponds to the condition (3.26) in the $N = 1$ superpotential, given by Eq. (3.1), reducing it to the form given by Eq. (3.27). Equation (3.30) shows that $N = 1$ gap vanishes at a singular point where a single massless dyon exists.

From the foregoing analysis it follows that a mass gap in the $N = 1$ confining phase of $SU(3)$ and $SO(6)$ theories vanishes when $N = 1$ parameters are properly tuned. As such, the nontrivial $N = 1$ fixed points in both these theories are exactly identified. It has been shown how to derive the curves for the Coulomb phase of these $N = 2$ Yang–Mills theories with classical gauge groups $SU(3)$ and $SO(6)$ by means of $N = 1$ confining phase superpotential. Transferring the critical points in $N = 2$ Coulomb phase to the $N = 1$ theories, we have found nontrivial $N = 1$ SCFT with the adjoint matter governed by a superpotential. It is speculated that this SCFT has a connection with the non-Abelian Coulomb phase of the Kutasov–Schwimmer model [24, 25].

REFERENCES

1. N. Seiberg and E. Witten, (1994). *Nucl. Phys. B* **426** 19.
2. N. Seiberg and E. Witten, (1994). *Nucl. Phys. B* **431** 484.
3. M. P. Singh and B. S. Rajput, (1999). *Ind. J. Phys.* **73A** 425.
4. M. P. Singh and B. S. Rajput, (1999). *Prog. Theor. Phys.* **102** 843.
5. M. P. Singh and B. S. Rajput, (1999). *Nuovo Cim.* **112A** (in press).
6. B. De Wittand A. Van Proeyen, (1984). *Nucl. Phys. B* **245** 89.
7. B. S. Rajput, H. C. Chandola, and J. M. S. Rana, (1991). *Phys. Rev. D* **43** 3550.
8. N. Seiberg, (1994). *Phys. Rev. D* **49** 6557.
9. N. Seiberg, (1994). *Phys. Rev. D* **50** 1092.
10. N. Seiberg, (1995). *Nucl. Phys. B* **435** 125.
11. S. Gukov and I. Polyubin, (1997). *Phys. Lett. B* **391** 115.
12. N. Seiberg, E. Witten, P. C. Argyres, and M. R. Plesser, (1996). *Nucl. Phys. B* **461** 71.
13. K. Intriligator and N. Seiberg, (1994). *Nucl. Phys. B* **431** 551.
14. P. C. Argyres and M. R. Douglas, (1995). *Nucl. Phys. B* **448** 93.
15. Seiji Terashima and Sung-Kil Yang, (1997). *Phys. Lett. B* **391** 107.
16. K. Intriligator, (1994). *Phys. Lett. B* **326** 409.
17. S. Elizur, A. Forge, A. Giveon, and E. Rabinovici, (1996). *Nucl. Phys. B* **459** 160.
18. D. Kutasov, A. Schwimmer, and N. Seiberg, (1996). *Nucl. Phys. B* **459** 455.
19. A. Klemm, W. Lerche, S. Yankielowicz, and S. Theisen, (1995). *Phys. Lett. B* **344** 169.

20. P. C. Argyres and A. E. Faraggi, (1995). *Phys. Rev. Lett.* **74** 3931.
21. T. Eguchi, K. Hori, K. Ito, and S. K. Yang, (1996). *Nucl. Phys. B* **471** 430.
22. A. Klemm, W. Lerche, and S. Theisen, (1996). *Int. J. Mod. Phys. A* **11** 1929.
23. A. Brandhuber and K. Landsteiner, (1995). *Phys. Lett. B* **358** 73.
24. D. Kutasov, (1995). *Phys. Lett. B* **351** 230.
25. D. Kutasov and A. Schwimmer, (1995). *Phys. Lett. B* **354** 315.